First order perturbative calculations for a conducting liquid jet in a solenoid
DRAFT 2

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December 9, 2000

Abstract

A perturbative calculation is given of the behavior of a continuous jet of conducting fluid as it enters and leaves a solenoidal magnetic field. It is assumed that the changes in direction, jet cross section and velocity are small.

If the jet enters the field along, or close to the axis, then the induced forces are compressive and retarding. The jet slows, suffers an increase in hydrostatic pressure, and increases in diameter; later, the jet re accelerates and shrinks. As the jet leaves the field, the hydrostatic pressure becomes negative and cavitation may occur.

If the jet enters at an angle to the axis, there are, in addition, deflections and elliptical deformations of the jet.

Formulae are given for these effects and numerical values given for the example of a solenoidal field with a Gaussian axial profile.

Contents

1 Introduction **2**

2 Formulae **2**

2.1 Introduction 2
2.2 Induced azimuthal current 2
2.3 Radial forces and Hydrostatic pressure 3
2.4 Axial force 4
2.5 Axial accelerations 5
2.6 Transverse forces and deflections 6
2.7 Induced axial current 7
2.8 Transverse elliptical distortion 8

3 Gaussian Case 9

4 Study 2 Example 10

5 Using Calculated Fields 15

6 Conclusion 15

1 Introduction

A mercury jet, injected at an angle respect to the axis of the solenoidal field, is the current baseline solution for the Feasibility Study II[1]. The interaction of the liquid-metal jet with the strong 20 T target solenoid has as result a number of forces on the jet which potentially may affect the viability of this target.[2],[3],[4] We present here perturbative calculations which confirm the findings of previous authors.

2 Formulae

2.1 Introduction

The jet is assumed to have an initial radius r, and be traveling at a velocity v. Changes in radius, shape, direction and velocity are all assumed to be small. The angle between the jet and solenoid axes is also assumed to be small.

No viscosity

$r << L$

In the following formulae, the coordinate system is defined by the jet; z is along the direction of motion and r is perpendicular to z. If the jet is along the axis then z is also the solenoid axis, and the azimuthal field $B_\phi = 0$.

If the jet is not directed along the solenoid axis, then we also define y in a direction perpendicular to z (the jet axis) and away from the solenoid axis; and x perpendicular to y and z.

$$r' = r + \frac{y\theta z}{r}$$

$$s' = s - y\theta$$

See fig. 2.1

2.2 Induced azimuthal current

The magnetic flux through a circle of radius r perpendicular to the jet axis is

$$\Phi = \int_S dS\vec{n} \cdot \vec{B} \approx \pi r^2 B_z(x, y, z) \tag{1}$$

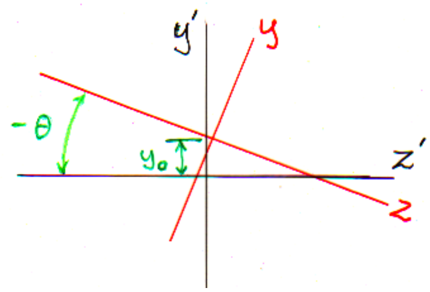


Figure 1: Schematic of the geometrical arrangement of solenoid and jet

As a liquid metal jet passes axially down such a field at a velocity $v = dz/dt$, a circumferential potential will be generated[6]

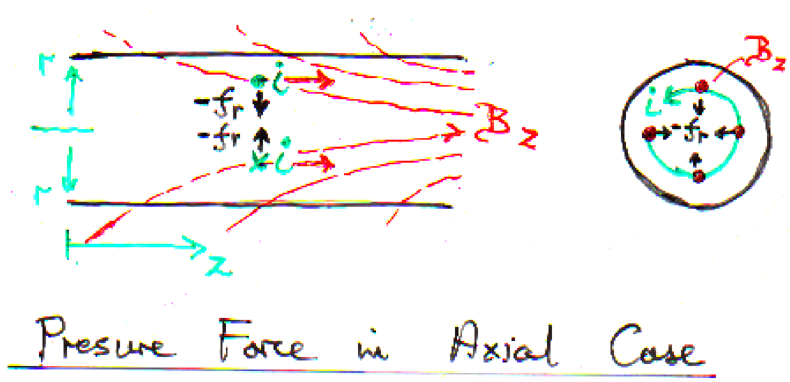
$$V \equiv \oint \vec{E} \cdot \vec{dl} = -\frac{d\Phi}{dt} = \pi r^2 v \frac{dB_z(x,y,z)}{dz} \tag{2}$$

If the metal electrical conductivity σ is low enough so that the resulting current has a negligible effect on the field, then the azimuthal current density i_ϕ will be

$$i_\phi \approx \frac{V}{2\pi r}\sigma \tag{3}$$

$$i_\phi \approx \frac{rv\kappa}{2}\frac{dB_z(0,0,z)}{dz} \tag{4}$$

2.3 Radial forces and Hydrostatic pressure



The induced radial force per unit volume ($dr\ r\ d\phi dz$) is

$$f_r = B_z i_\phi \approx \frac{r}{2}\ v\kappa\ B_z \frac{dB_z}{dz} \tag{5}$$

If we assume the effects of the fields are small so that the jet radius and liquid velocities do not vary by large fractions, and if we ignore radial inertia, then the hydrostatic pressures in a jet of outside radius r_o, at radius r, will be given by

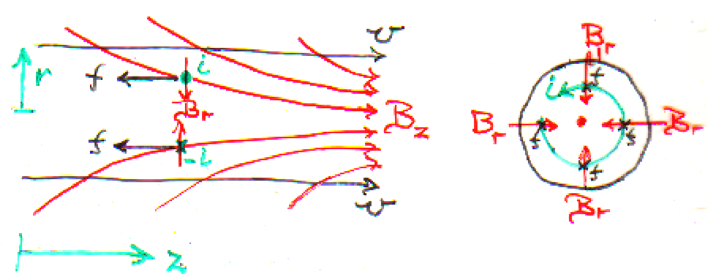
$$p(r,z) = \int_{r_o}^{r} -f_r dr \approx \left(\frac{r_o^2 - r^2}{4}\right) \; \upsilon\kappa \; B_z \frac{dB_z}{dz} \tag{6}$$

2.4 Axial force

The above hydrostatic pressure is a function of z, and gradients in it will exert axial pressures f_p on the liquid that must be added to the magnetic term f_z(hydrostatic).

$$f_p(\text{hydrostatic}) = \frac{dp(r,z)}{dz} \approx \; - \; \left(\frac{r_o^2 - r^2}{4}\right) \; \upsilon\kappa \; \frac{d}{dz}\left(B_z \frac{dB_z}{dz}\right) \tag{7}$$

To this must be added the axial forces induced directly by the fields acting on the asymuthal currents:



Retarding Force in Axial Case

If the jet is aligned with the field axis ($\theta = 0$, the radial field is given by

$$B_r(\theta = 0) \approx -\frac{r}{2}\frac{\partial B_z(0,0,z)}{\partial z} \tag{8}$$

The induced axial force per unit volume $(dr \; d\phi \; dz)$ is

$$f_z(\theta = 0) \;\; = \;\; f_p(r,z) \;\; - \;\; B_r i_\phi$$

If the jet is at an able to the magnetic axis, then there is an additional shear force:

$$f_z(\theta) \;\; = \;\; f_p(r,z) \;\; - \;\; B_y i_\phi \; \sin(\phi)$$

giving, in all:

$$f_z \approx -\left(\frac{r_o^2 - r^2}{4}\right) v\kappa \frac{d}{dz}\left(B_z \frac{dB_z}{dz}\right) + \frac{r^2}{4} v\kappa \left(\frac{dB_z}{dz}\right)^2 + \frac{r v\kappa}{2} B_y \frac{dB_z}{dz} \sin(\phi) \tag{9}$$

On the jet axis:

$$f_z \approx -\left(\frac{r_o^2}{4}\right) v\kappa \frac{d}{dz}\left(B_z \frac{dB_z}{dz}\right) \tag{10}$$

On the outer surface, averaged over the azymuthal angle ϕ, or in the absence of a B_y:

$$f_z \approx \frac{d}{dz}\left(B_z \frac{dB_z}{dz}\right) + \frac{r^2}{4} v\kappa \left(\frac{dB_z}{dz}\right)^2 \tag{11}$$

On the outer surface, with a finite B_y, as in the case of a jet at an angle to the magnetic axis:

$$f_z \approx \frac{d}{dz}\left(B_z \frac{dB_z}{dz}\right) + \frac{r_o^2}{4} v\kappa \left(\frac{dB_z}{dz}\right)^2 + \frac{y v\kappa}{2} B_y \frac{dB_z}{dz} \tag{12}$$

and the average force of the disk of radius r_o is given by integrating the terms

$$< f_z > \approx \frac{r_o^2}{8} v\kappa \left(\left(\frac{dB_z}{dz}\right)^2 + \frac{d}{dz}\left(B_z \frac{dB_z}{dz}\right)\right) \tag{13}$$

2.5 Axial accelerations

These forces will then decelerate, or accelerate layers of the fluid, thus inducing differences of liquid velocity as a function of radius

$$\frac{dv}{dz} = \frac{f}{\rho\ v} \tag{14}$$

$$\Delta\ v(r,z) = \int_{z_o}^{z} (f_z + f_p) \frac{1}{v\rho}\ dz \tag{15}$$

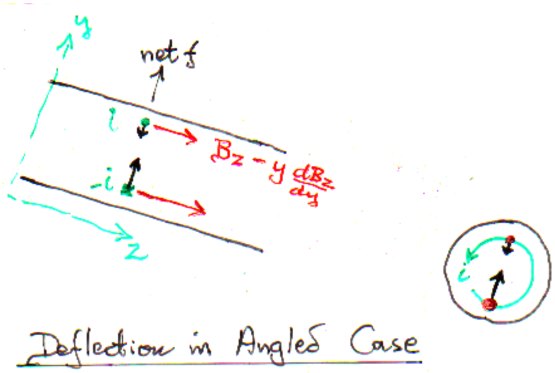
The average change in velocity is then

$$< \Delta v > (z) = \frac{\kappa}{\rho} \frac{r_o^2}{8} \left(\int_{z_o}^{z} \left(\frac{dB_z}{dz}\right)^2 + \frac{d}{dz}\left(B_z \frac{dB_z}{dz}\right)\ dz\right) \tag{16}$$

and the radius as a function of z is

$$r(z) = r_o \left(1 - \frac{< \Delta v > (z)}{v}\right) \tag{17}$$

2.6 Transverse forces and deflections



¿From above, the radial force per unit volume ($dr\ r\ d\phi dz$) is

$$f_r = B_z i_\phi \approx \frac{r}{2}\ \upsilon\kappa\ B_z \frac{dB_z}{dz} \tag{18}$$

If B_z varies with a transverse distance y, then the component of this radial force in the y direction is

$$f_y = f_r\ \sin\phi \tag{19}$$

and the net deflective force dF_y per unit length dz is

$$\frac{dF_y}{dz} = \int_o^r \int_o^{2\pi} \frac{r}{2}\ \upsilon\kappa\ \frac{dB_z}{dy}\ r \sin\phi^2 \frac{dB_z}{dz}\ r\ dr d\phi \tag{20}$$

$$= \frac{\upsilon\kappa}{2}\ \frac{dB_z}{dy}\ \frac{dB_z}{dz} \int_o^{2\pi} \sin\phi^2 d\phi \int_o^r r^3\ dr \tag{21}$$

$$= \frac{\pi}{8}\upsilon\kappa\ r^4\ \frac{dB_z}{dy}\ \frac{dB_z}{dz} \tag{22}$$

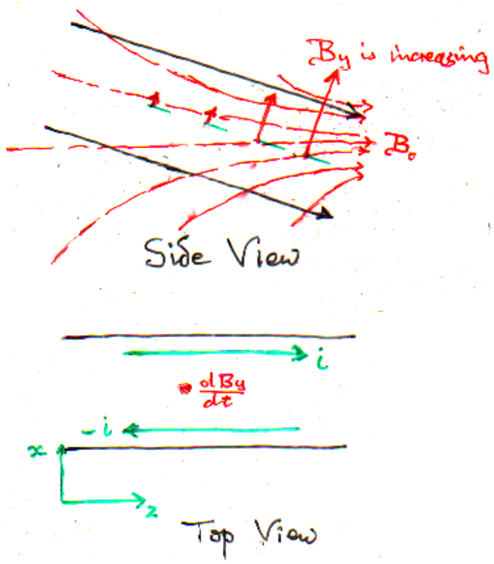
and the change in transverse velocity

$$\begin{aligned} \frac{dv_y}{dz} &= \frac{1}{v}\frac{dv_y}{dt} \\ &= \frac{\frac{dF_y}{dz}dz}{v\rho\pi r^2 dz} \\ &= \frac{\kappa r^2}{8\rho}\frac{dB_z}{dy}\frac{dB_z}{dz} \end{aligned}$$

and the inverse radius of bend is

$$\frac{d^2y}{dz^2} = \frac{d\theta}{dz} = \frac{\kappa\ r^2}{8\ v\ \rho}\ \frac{dB_z}{dy}\ \frac{dB_z}{dz} \tag{23}$$

2.7 Induced axial current



Consider a transverse field component B_y

The magnetic flux between transverse positions $-x$ to x and dz is

$$d\Phi_y = 2xdzB_y(z) \tag{24}$$

As a liquid metal jet passes axially down such a field at a velocity $v = dz/dt$, axial voltage gradients will be generated

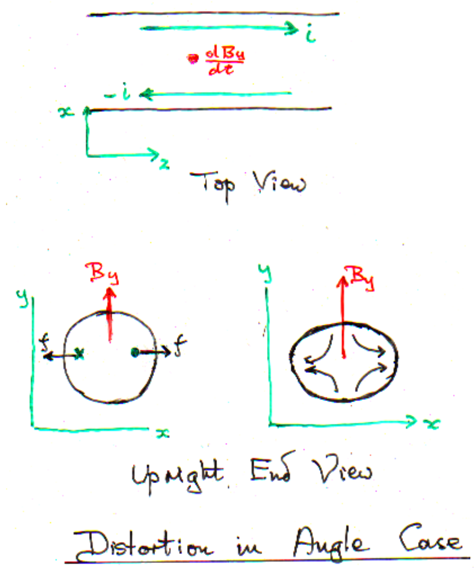
$$G = x\frac{dB_y}{dz}v \tag{25}$$

If the metal electrical conductivity κ is low enough so that the resulting current has a negligible effect on the field, then the axial current density i_z will be

$$i_z = G\sigma \tag{26}$$

$$i_z \ = xv\kappa\frac{dB_y(0,0,z)}{dz} \tag{27}$$

2.8 Transverse elliptical distortion



If the jet is not on the solenoid axis, the axial induced currents interacting with the transverse fields will generate distorting forces on the jet. These transverse forces per unit volume $dx\ dy\ dz$ are

$$f_x = i_z\ B_y \tag{28}$$

$$= x\ v\ \kappa\ B_y\ \frac{dB_y}{dz} \tag{29}$$

Ignoring surface tension, there is no restoring force, so the shape will change. The sides will go in and the height increase. A conservative estimate of the rate of width reduction can be obtained assuming that the only inertial resistance comes from the elements themselves. Then

$$\frac{d^2x}{dz^2} < \frac{f_x}{v\ \rho} = \frac{x\ \kappa}{\rho}\ B_y\ \frac{dB_y}{dz} \tag{30}$$

In reality the inertial constraint will be greater, because the inward motion of elements on the sides must generate equal outward motions at the top and bottom, and azymuthal motions between the two. Including these effects we can guess

$$\frac{d^2y}{dz^2} \approx \frac{1}{3}\ \frac{x\ \kappa}{\rho}\ B_y\ \frac{dB_y}{dz} \tag{31}$$

If this velocity is sufficiently small, then surface tension will restrain it. The total inward force on each side is

$$
\begin{aligned}
\frac{dF_B}{dz} &\approx \frac{1}{3}\; v\; \kappa\; B_y\; \frac{dB_y}{dz}\; \int_{-\pi}^{\pi}\int_{o}^{r} r\;\; \cos\phi dr\; r\; d\phi \\
&\approx \frac{2}{9}\; r^3\; v\; \kappa B_y \frac{dB_y}{dz}
\end{aligned}
$$

and this is opposed by the surface tension difference for a finite ellipticity $\epsilon = y/x$

$$
\frac{dF_T}{dz} \approx 2\; T\; \epsilon \tag{32}
$$

where T is the surface tension.

So the equilibrium ellipticity is

$$
\epsilon_{equilib} \approx \frac{1}{3}\; \frac{r^3}{3\;T}\; \kappa\; B_y\; \frac{dB_y}{dz} \tag{33}
$$

Again, this is only an order of magnitude calculation since the factor 1/3 is only an estimate for the inertial effects in distorting the ellipse. With the factor removed, however, the equation gives a harder maximum value.

3 Gaussian Case

We can consider a field that varies as a Gaussian in z The fields of the solenoid and coordinates are denoted with primes ($'$).

The Fields in the magnet system are:

$$
B'_z(r', z') \approx B_o e^{-\frac{z'^2}{2\sigma_z^2}} - \frac{1}{4} r'^2 \frac{\partial^2 B'_z(0, z')}{\partial z'^2} \tag{34}
$$

$$
B'_r(r', z') \approx -\frac{1}{2} r' \frac{\partial B'_z(0, z')}{\partial z'} \tag{35}
$$

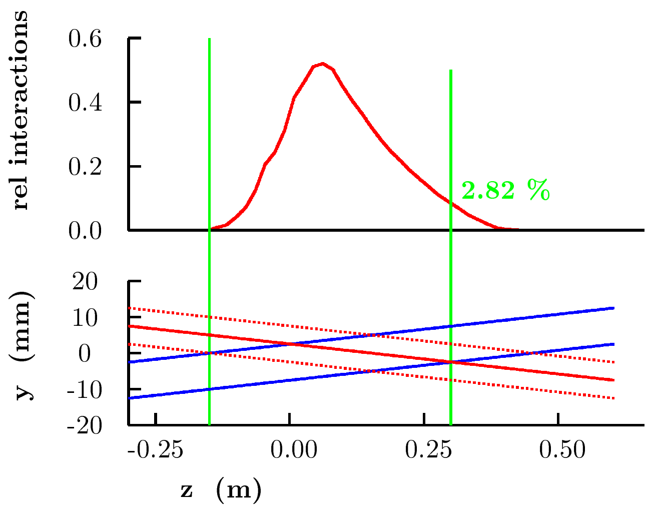
In the coordinate system of the jet, see Fig. 2.1, assuming a very small angle θ then $x' = x$, $y' = y + z\theta$, $r'^2 = r^2 + 2zy\theta$ and $z' = z - y\theta$ and the fields are

$$
\begin{aligned}
B_z(x,y,z) &\approx \left[B'_z(r', z') - \frac{1}{2}\theta y \frac{\partial B'_z(0, z')}{\partial z'} \right] \\
B_x(x,y,z) &\approx -\frac{1}{2} x \frac{\partial B'_z(0, z')}{\partial z'} \\
B_y(x,y,z) &\approx -\left[\frac{1}{2}(y + z\theta) \frac{\partial B'_z(0, z')}{\partial z'} - \theta B'_z(r', z') \right] \\
\frac{\partial B_z(x,y,z)}{\partial z} &\approx \left[\frac{\partial B'_z(0, z')}{\partial z'} - \frac{1}{2} y\theta \frac{\partial^2 B'_z(0, z')}{\partial z'^2} \right] \\
\frac{dB_y(x,y,z)}{dz} &\approx \left[\frac{1}{2}\theta \frac{\partial B'_z(0, z')}{\partial z'} \;-\; \frac{1}{2}(y + z\theta) \frac{\partial^2 B'_z(0, z')}{\partial z'^2} \right]
\end{aligned} \tag{36}
$$

4 Study 2 Example

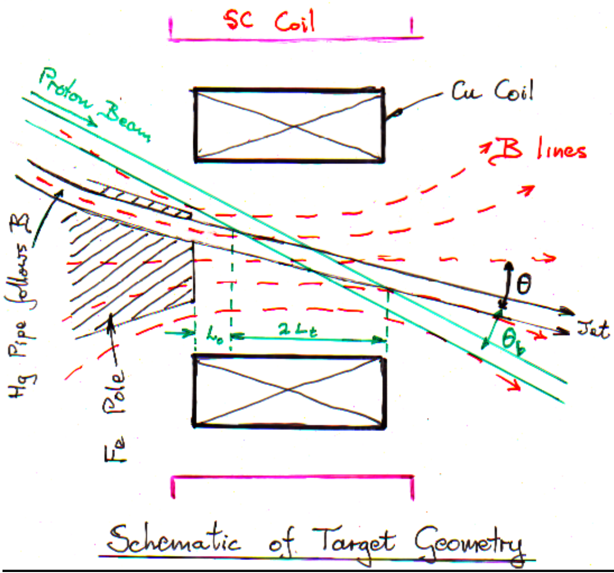
In the Study 2 target system, the beam with rms radius σ_r intersects a mercury jet of radius r_o at an angle $\theta_{crossing}$. The forward velocity of the jet is v_o. The intervals between pulses is t, and it will be assumed here that after a pulse, all the mercury outside of the nozzle is dispersed. The nozzle is at z_{nozzel} with respect to the intersection of the beam and jet center lines. Consider the following parameters:

σ_r	1.5	mm
r_o	5	mm
$\theta_{crossing}$	33	mrad
v_o	20	m/sec
t	20	ms
z_{nozzel}	-.375	m



The distribution of resulting interactions as a function of z is shown above. At the time of a second, or subsequent bunch, the newly established jet will extend a distance $z_{jet} = v_o\ t = 0.6$ m from the nozzle. It is seen that only 2.5 % of the interactions would occur after this location, had the beam extended indefinitely. Thus there is a negligible loss from this limited jet extent.

Thus the total length over which the jet must propagate without serious magnetic disruption is from the nozzle to a point 0.6 m downstream. In order to minimize the field non uniformity over this length, the magnetic center (approximate point of maximum B_z is placed at the center of this length. i.e. the magnetic center is set at a distance $z_{magnet} = z_{jet}/2 - z_{nozzel} = \ - .15$ m with respect to the jet-beam intersection. the figures shown here use a horizontal scale with $z = 0$ at the magnetic center.



A sketch of the layout is shown above. The proton beam enters at an angle θ_{beam} with respect to the magnet axis. The jet is at an angle $\theta_{jet} = \theta_{beam} - \theta_{crossing}$. The vertical distance y_o from the magnet center ($z = 0, r = 0$) to the jet axis at $z = 0$ can be chosen to minimize beam disruption. We assuming a Gaussian distribution of B'_z vs z', with a maximum value of B_o, The jet conductivity κ, density ρ, and surface tension $T_{surface}$, and the other parameters are given below:

B_o	20	T
σ'_z	.8	m
θ_{jet}	-67	mrad
y_o (without surface tension)	2	mm
y_o (with surface tension)	1	mm
κ	10^6	Ω m
ρ	13.5 10^4	kg/m^3
T_{surface}	.456	N/m
$p_{\text{gas}} = p_{\text{atmospheric}}$	10^5	N/m^2

Plots are shown for

a) The axial magnetic field B_z

b) The vertical magnetic field B_y

c) The rate of change of the transverse field B_y

c) The hydrostatic pressure on the jet axis with respect to the environment outside the jet ($p_{\mathrm{axis}} - p_{\mathrm{gas}}$)

d) The average deceleration of the jet Δ_v(Ave)

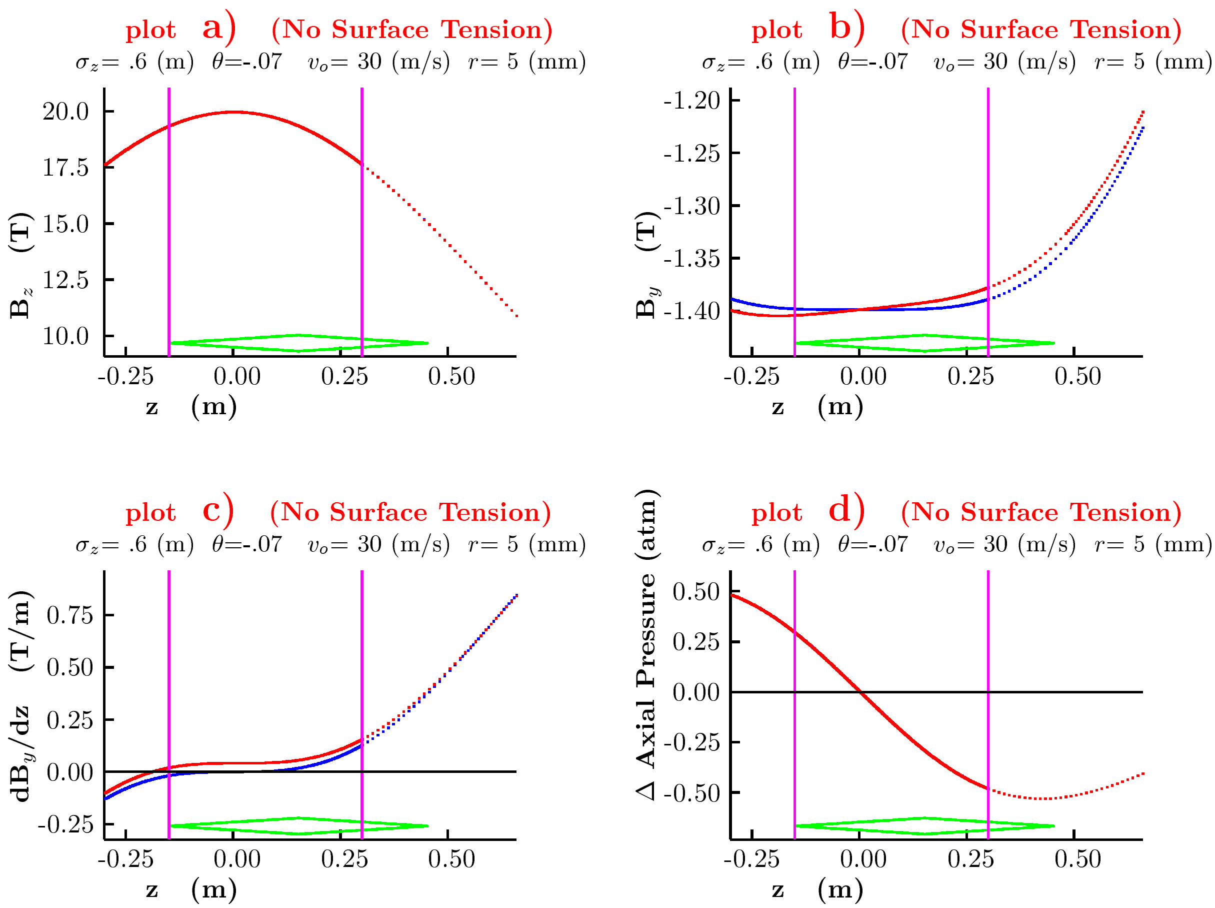
e) The maximum shear acceleration/deceleration of the upper/lower limits of the jet Δ_v(shear)

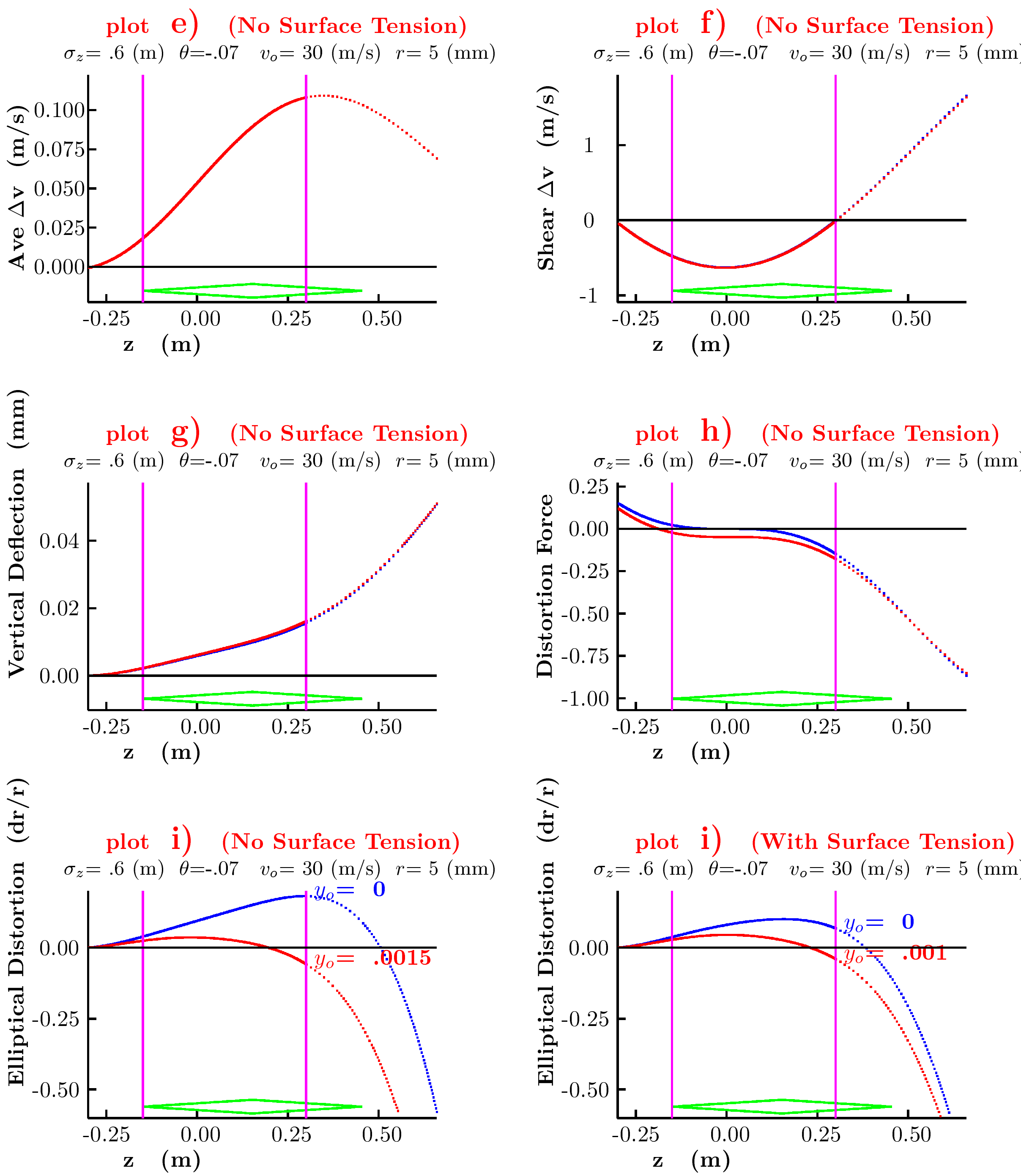
f) The vertical displacement of the jet due to deflecting forces y

g) The total transverse distorting forces F_{distort}

h) The resulting elliptical distortion (δx/r=$-\delta$y/r), without surface tension

1. The resulting elliptical distortion (δx/r=$-\delta$y/r), without surface tension





We see that over the extent of the new jet (from - .3 to .3 m):

- The maximum axial field deviations are $+/-$ 1.25 T = 6%
- The transverse fields seen by the jet are approximately 1.4 T, and vary at the ends by only up to $+/-$ 0.01 T = $+/-$ 0.7 %. But it is this small variation of transverse field that generate axial current in the jet, leading to jet shape distortions that are the most serious constraint on the magnet design.
- The axial pressure difference has a minimum of - 0.5 atmospheres. Thus if the jet is operating in a gas (He or Argon) at a pressure of 1 atmosphere, then the minimum pressure will be + 0.5 atmospheres, and there will be no tendency to cavitate prior to the arrival of the beam.
- The maximum average deceleration of the jet is small compared to the average jet velocity: 0.11/30≈0.3%.
- The maximum decelerations (from shear forces) are also small compared to the average jet velocity: 0.7/30≈2.3%.
- The deflections of the jet are very small: 20 μm.
- The distorting Forces, in the absence of surface tension, are of the same order of magnitude as the surface tension: .15 N/m compared with the surface tension of .5 N/m.
- The resulting calculated jet distortions (Δ width / ave width) with no jet axis displacement ($y_o = 0$) are 18% without surface tension, and 7%, with surface tension. Both can be reduced by small jet axis displacements: to 6% without surface tension, and 4% with surface tension.

These disruptions are all relatively small, and should, with the fields assumed, cause no problems. But relatively small field changes can make the beam distortions uncomfortably large. It thus appears that considerations of these distortions will set the tolerances on the magnetic field shape. Tuning of these distortions can be achieved by vertical displacements of the jet, and provision for mechanically adjusting this should be considered.

5 Using Calculated Fields

Plots of the magnetic fields and derivatives needed in this calculations including the relative tilded of $\theta = 0.1$ between jet and solenoid axis are shown in Figs.**??** and **??**

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Compared with a jet velocity of 20 m/sec, these are all relatively low. Earlier optimizations had called for a jet radius of 1 cm, which gave values of Δv 4 times higher, and thus more significant.

6 Conclusion

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References

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[2] K. McDonald (Ed), *An R&D Program for Targetry and Capture at a Muon-Collider Source*, Proposal to the BNL AGS Division, p.23

[3] R. Weggel, *Behavior of conducting solid or liquid jet moving in magnetic field: 1) Paraxial; 2) Transverse; 3) Oblique*, BNL report BNL-65611/CAP-220-Muon-98R

[4] **Phys. Today**, Feb 2000, p.29

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[6] K. McDonald, *Damping and Radial Pinching Effects*,

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